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The flow and tension spaces and lattices of signed graphs

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Abstract

This paper is to introduce circuit, bond, flow, and tension spaces and lattices for signed graphs, and to study the relations among these spaces and lattices. The key ingredient is to introduce circuit and bond characteristic vectors so that the desired spaces and lattices can be defined such that their dimensions and ranks match well to that of matroids of signed graphs. The main results can be stated as follows: (1) the classification of minimal directed cuts; (2) the circuit space (lattice) equals flow space (lattice), and the bond space equals the tension space; (3) the bond lattice equals the row lattice of the incidence matrix, and the reduced bond lattice equals the tension lattice; and (4) for unbalanced signed graphs, the module of potentials is isomorphic to the module of tensions if the coefficient ring is 2-torsion free.

1. Introduction

A signed graph is a graph (loops and multiple edges are allowed) whose edges are given either a positive sign $+1$ or a negative sign -1, but not both. An ordinary graph can be considered as a signed graph whose edges are all given positive sign. Guided by matroid theory, Zaslavsky [10,11] generalized the notions of circuit, bond, orientation of graphs to signed graphs, and the notions of directed circuit and bond of directed graphs to oriented signed graphs. It is well known in graph theory that the cycle and bond spaces are orthogonal complements each other; see, for example, the books of Bollabás [4, Theorem 9, p. 43], Bondy and Murty [5, Chapter 12], and Godsil and Royle [7, Chapter 14]. Moreover, the lattices of such spaces are integral spans of circuit and bond characteristic vectors, respectively. It should be interesting and valuable to

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extend these fundamental results on graphs to signed graphs, for signed graphs appear naturally in many other fields, ranging from theory to application. This extension seems to be not very difficult, for most related concepts already exist in literature; however, it is by no means obvious. Searching the literature on the subject carefully, we found no references to such desired spaces and lattices and the relations holding among them, to the best of our knowledge. The present paper is to introduce circuit and bond spaces and lattices for signed graphs, and to extend the fundamental theorems on such spaces and lattices for ordinary graphs to signed graphs.

We choose definitions of circuit, bond, and orientation as given by Zaslavsky [10,11] for signed graphs. Further to these known concepts, we first introduce the notions of cut, directed cut, and minimal directed cut; bond is another name for minimal cut. Section 2 is to clarify the relation holding among these concepts; and to characterize minimal directed cuts in explicit description. Next, we define *indicator functions* I_C and I_B for circuits C and bonds B by modifying the ordinary characteristic functions with weight. Then we define a *coupling* $[\varepsilon_1, \varepsilon_2]$ for any pair of orientations ε_1 and ε_2 on signed subgraphs. For an oriented signed graph (Σ, ε), the coupling enables us to define *characteristic vectors* $[\varepsilon_C, \varepsilon]I_C$ and $[\varepsilon_B, \varepsilon]I_B$ for directed circuits (C, ε_C) and directed bonds (B, ε_B). The integral spans of such vectors are then called the *circuit lattice* $Z(\Sigma, \varepsilon; \mathbb{Z})$ and *bond lattice* $B(\Sigma, \varepsilon; \mathbb{Z})$, respectively. Lastly, the *flow lattice* $F(\Sigma, \varepsilon; \mathbb{Z})$ and *tension lattice* $T(\Sigma, \varepsilon; \mathbb{Z})$ are introduced for signed graphs, similar to homology and cohomology groups of graphs with integral coefficients. The *flow space* $F(\Sigma, \varepsilon; \mathbb{R})$ and *tension space* $T(\Sigma, \varepsilon; \mathbb{R})$ are defined analogously with real coefficients.

Let Σ be a signed graph with an orientation ε. The main results of the paper can be stated as follows:

1. The characterization of minimal directed cuts;
2. $Z(\Sigma, \varepsilon; \mathbb{Z}) = F(\Sigma, \varepsilon; \mathbb{Z})$ and $\tilde{B}(\Sigma, \varepsilon; \mathbb{Z}) = T(\Sigma, \varepsilon; \mathbb{Z})$;
3. $F(\Sigma, \varepsilon; \mathbb{R})$ and $T(\Sigma, \varepsilon; \mathbb{R})$ are orthogonal complements in $\mathbb{R}^{E(\Sigma)}$;
4. $2\tilde{B}(\Sigma, \varepsilon; \mathbb{Z}) \subseteq B(\Sigma, \varepsilon; \mathbb{Z}) = \operatorname{Row} \boldsymbol{M}$;
5. $R^{b(\Sigma)} \oplus T(\Sigma, \varepsilon; R) \simeq R^{V(\Sigma)}$;

where $V(\Sigma)$ and $E(\Sigma)$ are vertex and edge sets of Σ respectively, $b(\Sigma)$ is the number of balanced components of Σ, $\boldsymbol{M}$ is the incidence matrix of (Σ, ε), $\tilde{B}(\Sigma, \varepsilon; \mathbb{Z})$ is the reduced bond lattice, and R is a commutative ring having the unity 1 and the element 2 invertible.

There are plenty of references on signed graphs; see the survey by Zaslavsky [13]. Related to the present paper are the work of Zaslavsky [10], Bouchet [6], Khelladi [8], and the recent work by Beck and Zaslavsky [1,2]. For further information about signed graphs, we refer to Zaslavsky's papers [9–12].

2. Characterization of (minimal directed) cuts

Let us recall briefly some known concepts and notations of signed graphs that we shall need in the present paper. These concepts and notations may be found in [10], except the notion of (minimal directed) cuts and their characterization; see Proposition 2.1 and Theorem 2.4 below. Throughout the whole paper let $\Sigma = (V, E, \sigma)$ be a signed graph, where V is the vertex set, E is the edge set, and $\sigma : E \to \{\pm 1\}$ is the sign function. For a vertex subset $X \subseteq V$, we denote by $E(X)$ the set of edges whose end vertices are contained in X, and by $\Sigma(X)$ the signed subgraph $(X, E(X), \sigma|_{E(X)})$. For an edge subset $S \subseteq E$, we denote by $\Sigma(S)$ the signed subgraph $(V, S, \sigma|_S)$.

A *cycle* of Σ is a simple closed path; the *sign* of a cycle is the product of signs of all edges. A cycle is said to be *balanced* if its sign is positive; and *unbalanced* otherwise. A signed graph is said to be *balanced* if its every cycle is balanced; and *unbalanced* otherwise. The connected components of Σ are partitioned into two types: balanced components and unbalanced components. We denote by $b(\Sigma)$ the number of balanced components of Σ.

A *circuit* C of Σ is either a balanced cycle, said to be of *Type I*; or an edge set consisting of two unbalanced cycles C_1, C_2, whose vertex sets have exactly one common vertex, written $C = C_1C_2$ and said to be of *Type II*; or an edge set consisting of two unbalanced cycles C_1, C_2, and a simple path P (called *circuit path*) with at least one edge, written $C = C_1PC_2$ and said to be of *Type III*, such that $V(C_1) \cap V(C_2) = \emptyset$, and $(V(C_1) \cup V(C_2)) \cap V(P)$ contains exactly the initial and end vertices of the path P.

A *cut* of Σ is a non-empty edge subset of the form $U = [X, X^c] \cup E_X$, where $X \subseteq V(\Sigma)$ is non-empty, $[X, X^c]$ is the set of edges between X and its complement X^c, and $E_X \subseteq E(X)$ is minimal to have $\Sigma(E(X) - E_X)$ balanced, i.e., $\Sigma(E(X) - E_X + e)$ is unbalanced for any $e \in E_X$. A cut is said to be *minimal* if it does not properly contain any cut. A minimal cut is called a *bond*. A bond $B = [X, X^c] \cup E_X$ is said to be of *Type I* if $E_X = \emptyset$, and of *Type II* if $X = V$, and of *Type III* otherwise; the set E_X is called the *bond core* of B. Notice that a cut may not be a disjoint union of bonds. It is clear that if Σ is balanced and connected, its every cut has the form $[X, X^c]$. The following proposition characterizes cuts of an unbalanced signed graph.

Proposition 2.1. *Let Σ be a connected unbalanced signed graph.*

(a) *Let $U = [X, X^c] \cup E_X$ be a cut. If $\Sigma(X)$ is connected, so is $\Sigma(X) - E_X$.*
(b) *The removal of a cut increases the number of balanced components.*
(c) *A cut $B = [X, X^c] \cup E_X$ is a bond if and only if $\Sigma(X)$ is connected, and every component of $\Sigma(X^c)$ is unbalanced.*
(d) *An edge subset B is a bond if and only if B is minimal, in the sense that its removal increases the number of balanced components by one, i.e., $b(\Sigma - B) = b(\Sigma) + 1$ and $b(\Sigma - B + e) = b(\Sigma)$ for any $e \in B$.*

Proof. (a) Suppose $\Sigma(X) - E_X$ is disconnected. Take an edge e between any two connected components of $\Sigma(X) - E_X$. Then $e \in E_X$ and $\Sigma(X) - E_X + e$ is still balanced. This is a contradiction.

(b) It is trivial by definition of cut.

(c) The sufficiency is trivial. As for the necessity, let $B = [X, X^c] \cup E_X$ be a bond. If $X = V$, then $\Sigma(X) = \Sigma$ and $B = E_X$; it is obvious that $\Sigma(X)$ is connected and every component of $\Sigma(X^c)$ is unbalanced, since Σ is connected and $\Sigma(X^c)$ has no component. If $X \neq V$, we first claim that $\Sigma(X)$ is connected. Suppose $\Sigma(X)$ is decomposed into at least two connected components Σ_i. Set $X_i = V(\Sigma_i)$, $E_{X_i} = E(X_i) \cap E_X$, and $B_i = [X_i, X_i^c] \cup E_{X_i}$. Then B is decomposed into a disjoint union of smaller cuts B_i. This is a contradiction.

Next we claim that every component of $\Sigma(X^c)$ is unbalanced. Suppose $\Sigma(X^c)$ has at least two components and one of them is balanced, say Σ_0; then $U_0 = [X_0, X_0^c]$ with $X_0 = V(\Sigma_0)$ is a smaller cut that is properly contained in B; again this is a contradiction. Suppose $\Sigma(X^c)$ is connected and balanced. If the edges of $[X, X^c]$ are all positive or all negative, then $U' = [V, \emptyset] \cup E'_V$ with $E'_V = E_X$ being a smaller cut; this is a contradiction. If $[X, X^c]$ contains both positive and negative edges, then $U' = [V, \emptyset] \cup E'_V$ is a cut, where E'_V is the union of E_X and the set of negative edges of $[X, X^c]$. The cut U' is properly contained in B; this is a contradiction. Thus every component of $\Sigma(X^c)$ is unbalanced.

(d) This is contained implicitly in [11]. The necessity is trivial by Part (c). As for sufficiency, let B be a minimal edge set of Σ such that $b(\Sigma - B) = b(\Sigma) + 1$. Then all components of

$\Sigma(E - B)$ are unbalanced except for exactly one balanced component Σ_0. Set $X = V(\Sigma_0)$ and $E_X = E(X) - E(\Sigma_0)$. Then $B = [X, X^c] \cup E_X$, and it is a bond by Part (c). □

Let $x = uv$ be an edge of Σ with end vertices u and v ($u = v$ if x is a loop). We denote by End(x) the multi-set $\{u, v\}$, i.e., End(x) $= \{u, v\}$ if x is a loop. An *orientation* of x is a multi-set $\{\varepsilon(u, x), \varepsilon(v, x)\}$ of two elements over the set $\{\pm 1\}$, such that

$$\varepsilon(u, x)\varepsilon(v, x) = -\sigma(x);$$

we write $\varepsilon_x = \{\varepsilon(u, x), \varepsilon(v, x)\}$. Pictorially, an orientation of an edge $x = uv$ can be considered as two arrows assigned to x, each at its end vertices u and v, in such a way that the arrow points away from u if $\varepsilon(u, x) = 1$ and the arrow points towards u if $\varepsilon(u, x) = -1$. An edge x together with an orientation ε_x is called an *oriented edge*. Every edge other than a positive loop has exactly two orientations, while a positive loop has only one orientation.

An *orientation* of a signed graph $\Sigma = (V, E, \sigma)$ is an assignment where each edge of Σ is given an orientation. Alternatively, an orientation of Σ can be considered as a function $\varepsilon : V \times E \to \{0, \pm 1\}$ such that (i) $\varepsilon(u, x) = 0$ if $u \notin$ End(x), (ii) $\varepsilon(u, x)$ is assigned two opposite values if x is a positive loop with $u \in$ End(x), and (iii)

$$\varepsilon(u, x)\varepsilon(v, x) = -\sigma(x), \quad x = uv. \tag{2.1}$$

So ε is actually *not* a function, but a multi-valued function. A signed graph Σ with an orientation ε in the present paper may be considered a bi-directed ordinary graph of [6]; we call it an *oriented signed graph*, denoted by (Σ, ε).

Let $\nu : V \to \{\pm 1\}$ be a function, called a *switching function*. For the oriented signed graph $(\Sigma, \varepsilon) = (V, E, \sigma, \varepsilon)$, we define a sign function σ^ν and an orientation ε^ν by

$$\sigma^\nu(x) = \nu(u)\sigma(x)\nu(v), \qquad \varepsilon^\nu(u, x) = \nu(u)\varepsilon(u, x), \quad x = uv.$$

Indeed, $\varepsilon^\nu(u, x)\varepsilon^\nu(v, x) = -\sigma^\nu(x)$. So ε^ν is an orientation of the signed graph (V, E, σ^ν). A *switching* from $(V, E, \sigma, \varepsilon)$ to $(\Sigma^\nu, \varepsilon^\nu) = (V, E, \sigma^\nu, \varepsilon^\nu)$ by a switching function ν is the replacement of σ by σ^ν and ε by ε^ν. Notice that switching does not change the balance of cycles.

Let W be a walk of Σ. We write W as a vertex-edge sequence:

$$W = u_0 x_0 u_1 x_1 \cdots u_n x_n u_{n+1},$$

where the edge x_i is incident with vertices u_i and u_{i+1}. The *sign* of W is the product

$$\sigma(W) := \prod_{i=0}^{n} \sigma(x_i).$$

A *direction* of W is a function ε_W with values ± 1, defined for the pairs (u_i, x_i) and (u_{i+1}, x_i), such that

$$\varepsilon_W(u_i, x_i)\varepsilon_W(u_{i+1}, x_i) = -\sigma(x_i), \qquad \varepsilon_W(u_i, x_{i-1}) + \varepsilon_W(u_i, x_i) = 0.$$

Every walk has exactly two directions. A walk W together with a direction ε_W is called a *directed walk*, denoted (W, ε_W). If a directed walk (W, ε_W) is closed, i.e., $u_0 = u_{n+1}$, then it is easy to see that

$$\varepsilon_W(u_{n+1}, x_n) = \begin{cases} -\varepsilon_W(u_0, x_0) & \text{if } W \text{ is positive,} \\ \varepsilon_W(u_0, x_0) & \text{if } W \text{ is negative.} \end{cases}$$

A directed closed positive walk is said to be *minimal* (or *irreducible*) if it does not properly contain any directed closed positive sub-walk.

A *direction* of a circuit C is an orientation ε_C on the signed subgraph $\Sigma(C)$ such that every vertex is neither a source nor a sink. A circuit C with a direction ε_C is called a *directed circuit*, denoted (C, ε_C). We automatically extend a direction ε_C to $V \times E$ by setting $\varepsilon_C(v, y) = 0$ when v and y are not incident in C. It is easy to see that a minimal directed closed positive walk can be constructed on any directed circuit along its direction, using the edges on the circuit path twice.

Lemma 2.2. *Let $U = [X, X^c] \cup E_X$ be a cut of Σ. Then there exists a switching function ν such that $\nu|_{X^c} \equiv 1$, $\sigma^\nu|_{E(X)-E_X} \equiv 1$, and $\sigma^\nu|_{E_X} \equiv -1$.*

Proof. Since $\Sigma(X) - E_X$ is balanced, there exists a switching function ν with $\nu|X^c \equiv 1$, such that all edges of $\Sigma^\nu(X) - E_X$ are positive. We claim that all edges of E_X are negative in Σ^ν. Suppose one edge $e \in E_X$ is positive. Clearly, $\Sigma^\nu(X) - E'_X$ is still balanced with $E'_X = E_X - \{e\}$. Then E_X is not minimal and U cannot be a cut. This is a contradiction. $\square$

A *direction* of a cut $U = [X, X^c] \cup E_X$ is an orientation ε_U on the signed graph $\Sigma(U)$ such that there exists a switching ν_X satisfying the following conditions:

$$\nu|_{X^c} \equiv 1, \qquad \sigma^{\nu_X}|_{E(X)-E_X} \equiv 1, \qquad \sigma^{\nu_X}|_{E_X} = -1, \qquad \varepsilon_U^{\nu_X}(u, x) = 1,$$

where $x = uv \in U$ with $u \in X$. We automatically extend ε_U to $V \times E$ by setting $\varepsilon_U(v, y) = 0$ if v and y are not incident in such that $v \in X$ and $y \in U$. We call the orientation $\varepsilon_U^{\nu_X}$ a *positive direction* of U in the signed graph Σ^{ν_X}. A cut U with a direction ε_U is called a *directed cut*, denoted (U, ε_U). We say that a cut U is *directed* in (Σ, ε) (or a *directed cut* of (Σ, ε)) if (U, ε) is a directed cut. Lemma 2.2 shows that any cut can be equipped with a direction. A directed cut (U, ε) with $U = [X, X^c] \cup E_X$ is said to be *minimal* if it does not properly contain any directed cut (U_1, ε) with $U_1 = [X_1, X_1^c] \cup E_{X_1}$ such that $U_1 \subseteq U$ and $E_{X_1} \subseteq E_X$. Notice that a minimal directed cut needs not be a directed bond, i.e., it may contain properly a cut. For example, the oriented singed graph in Fig. 1 is a minimal directed cut, but is not a bond; it contains two bonds $\{x_1, y_1, y_2, y_3, y_4\}$ and $\{x_2, y_1, y_2, y_3, y_4\}$.

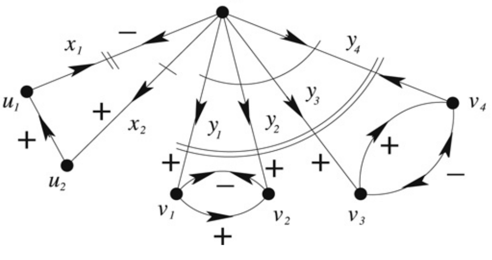

Fig. 1. A minimal directed cut that is not a bond.

Lemma 2.3. *Let (U, ε) be a directed cut, where $U = [X, X^c] \cup E_X$. If (U, ε) is a minimal directed cut, then $\Sigma(X) - E_X$ is connected.*

Proof. Let $\Sigma_1, \ldots, \Sigma_k$ be connected components of $\Sigma(X)$. Set $X_i = V(\Sigma_i)$, $E_{X_i} = E(X_i) \cap E_X$, and $U_i = [X_i, X_i^c] \cup E_{X_i}$. Then (U, ε) is a disjoint union of the directed cuts (U_i, ε_U). So, if

(U, ε) is minimal directed cut, then $\Sigma(X)$ must be connected. By Proposition 2.1(a), $\Sigma(X) - E_X$ is connected. □

Theorem 2.4. *Let Σ be a connected unbalanced signed graph. Let (U, ε) be a directed cut, where $U = [X, X^c] \cup E_X$. Then (U, ε) is a minimal directed cut if and only if*

(1) *(U, ε) is a directed bond; or*
(2) *$X \neq V$, $\Sigma(X) - E_X$ is connected, $\Sigma(X^c)$ contains at least one balanced component, and for each such balanced component Σ_0 of $\Sigma(X^c)$ the signed subgraph $\Sigma_0' := \Sigma(X \cup V(\Sigma_0)) - E_X$ is unbalanced.*

Proof. "⇒": Note that $\Sigma(X)$ is connected by Lemma 2.3. If $X = V$, then $U = E_X$; and in this case (U, ε) is a directed bond of Type II. This belongs to case (1). If $X \neq V$, then $\Sigma(X^c)$ is non-empty and is decomposed into disjoint components. When all components of $\Sigma(X^c)$ are unbalanced, it follows from Proposition 2.1(c) that (U, ε) is a directed bond, which is of case (1).

Otherwise, we have $X \neq V$, $\Sigma(X^c)$ contains at least one balanced component, and $\Sigma(X) - E_X$ is connected (by Lemma 2.3). (Of course, $\Sigma(X^c)$ may or may not contain any unbalanced component.) Let Σ_0 be a balanced component of $\Sigma(X^c)$. Suppose the signed subgraph $\Sigma_0' := \Sigma(X \cup V(\Sigma_0)) - E_X$ is balanced. Set $X' = V(\Sigma_0')$, $E_{X'} = E_X$, and $U' = [X', X'^c] \cup E_{X'}$. Then (U', ε) is a directed cut, and $[X', X'^c]$ is properly contained in $[X, X^c]$. This is contrary to the fact that (U, ε) is a minimal directed cut.

"⇐": It is clear by definition that a directed bond is a minimal directed cut.

Let (U, ε) be a directed cut satisfying the condition (2) in Theorem 2.4. We need to show that (U, ε) is a minimal directed cut. Suppose (U_1, ε) is a directed cut such that $U_1 \subsetneq U$ and $E_{X_1} \subseteq E_X$, where $U_1 = [X_1, X_1^c] \cup E_{X_1}$. Since U_1 contains less edges than U, each component of $\Sigma - U_1$ is a union (as vertex sets) of some components of $\Sigma - U$. In particular, $\Sigma(X_1) - E_{X_1}$ is a union of $\Sigma(X) - E_X$ and some other components of $\Sigma - U$. We first claim that $E_{X_1} = E_X$. Otherwise, if $e \in E_X - E_{X_1}$, then $\Sigma(X) - E_X + e$ is unbalanced; so $\Sigma(X_1) - E_{X_1}$ is unbalanced; this is a contradiction. Clearly, $\Sigma(X_1) - E_{X_1}$ does not contain any unbalanced component of $\Sigma - U$. Thus $\Sigma(X_1) - E_{X_1}$ must contain a balanced component Σ_0 of $\Sigma - U$; consequently, the unbalanced signed subgraph $\Sigma(X \cup V(\Sigma_0)) - E_X$ is contained in $\Sigma(X_1) - E_{X_1}$; so $\Sigma(X_1) - E_{X_1}$ is unbalanced; this is a contradiction. □

Note. As an immediate consequence of Theorem 2.4, the underlying cut of any minimal directed cut, other than a bond, contains at least one bond, but cannot be a disjoint union of bonds.

Theorem 2.5. *Every directed cut is a disjoint union of minimal directed cuts.*

Proof. Let $U = [X, X^c] \cup E_X$ be a cut with a positive direction ε_U. If $E_X = \emptyset$, then (U, ε_U) is an ordinary cut and the conclusion follows from the fact of ordinary directed cuts. So we may assume that $E_X \neq \emptyset$ and $\Sigma(X)$ is unbalanced. Let $\Sigma_1, \ldots, \Sigma_k$ be connected components of $\Sigma(X)$. Set $X_i = V(\Sigma_i)$, $E_{X_i} = E(X_i) \cap E_X$, and $U_i = [X_i, X_i^c] \cup E_{X_i}$. Then (U_i, ε_U) are directed cuts, and (U, ε_U) is a disjoint union of (U_i, ε_U). Now we may further assume that $\Sigma(X)$ is connected. Then $\Sigma(X) - E_X$ is connected by Lemma 2.3. Let $\Sigma_j' (1 \leq j \leq l)$ be balanced components of $\Sigma(X^c)$ such that $\Sigma(X \cup V(\Sigma_j')) - E_X$ are balanced. Set $X_j' = V(\Sigma_j')$, $U_j' = [X_j', X]$; and $Y = X^c - \bigcup_{j=1}^{l} X_j'$, $U' = [X, Y] \cup E_X$. Then (U, ε_U) is a disjoint union of directed cuts (U_j', ε_U) and (U', ε_U). It easy to see that these directed cuts have the form of minimal directed cuts in Theorem 2.4. □

3. The circuit and bond spaces

For the convenience of studying spaces associated to signed graphs, we introduce a coupling function for any two orientations on signed subgraphs of a signed graph Σ. Let ε_i $(i = 1, 2)$ be orientations on signed subgraphs Σ_i of Σ. The *coupling* of ε_i is a function $[\varepsilon_1, \varepsilon_2] : E \to \mathbb{Z}$, defined for each edge $x = uv$ by

$$[\varepsilon_1, \varepsilon_2](x) = \begin{cases} 1 & \text{if } x \in E(\Sigma_1) \cap E(\Sigma_2), \varepsilon_1(u, x) = \varepsilon_2(u, x), \\ -1 & \text{if } x \in E(\Sigma_1) \cap E(\Sigma_2), \varepsilon_1(u, x) \neq \varepsilon_2(u, x), \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

It is easy to verify that switching does not change coupling, i.e., $[\varepsilon_1^\nu, \varepsilon_2^\nu] = [\varepsilon_1, \varepsilon_2]$ for any switching function ν.

Let R be a commutative ring. Let R^E be the commutative ring of all functions from E to R. There is an obvious pairing $\langle\, , \rangle : R^E \times R^E \to R$, defined by

$$\langle f, g\rangle = \sum_{x\in E} f(x)g(x).$$

Let $(\Sigma, \varepsilon) = (V, E, \sigma, \varepsilon)$ be an oriented signed graph. Let C be a circuit of Σ with a direction ε_C. Viewing both (Σ, ε) and (C, ε_C) as oriented signed subgraphs of Σ, we have a coupling function $[\varepsilon, \varepsilon_C]$. The *circuit indicator* of C is a function $I_C : E \to \mathbb{Z}$, defined by

$$I_C(x) = \begin{cases} 2 & \text{if } x \in C \text{ and is on the circuit path,} \\ 1 & \text{if } x \in C \text{ and is not on the circuit path,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The product function $[\varepsilon, \varepsilon_C]I_C$ determines a vector in the Euclidean space $\mathbb{R}^E$ of real-valued functions on E, called the *characteristic vector* of the directed circuit (C, ε_C) for (Σ, ε). The *circuit space* (*lattice*) of (Σ, ε) is the real (integral) span of the characteristic vectors of all directed circuits of (Σ, ε), denoted $Z(\Sigma, \varepsilon; \mathbb{R})$ $(Z(\Sigma, \varepsilon; \mathbb{Z}))$.

Let (U, ε_U) be a directed cut of Σ, where $U = [X, X^c] \cup E_X$ with switching function ν_X, i.e., the edges of E_X are negative in Σ^{ν_X}, the edges of $\Sigma^{\nu_X}(X) - E_X$ are positive, and $\varepsilon(u, x) = 1$ for all $x = uv$ with $x \in U$ and $u \in X$. The *cut indicator* of U is a function $I_U : E \to \mathbb{Z}$, defined by

$$I_U(x) = \begin{cases} 2 & \text{if } x \in E_X, \\ 1 & \text{if } x \in [X, X^c], \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

The product function $[\varepsilon, \varepsilon_U]I_U$ determines a vector in $\mathbb{R}^E$, called the *characteristic vector* of the directed cut (U, ε_U) for (Σ, ε). The *bond space* (*lattice*) of (Σ, ε) is the real (integral) span of the characteristic vectors of all directed cuts of (Σ, ε), denoted $B(\Sigma, \varepsilon; \mathbb{R})$ $(B(\Sigma, \varepsilon; \mathbb{Z}))$. For a bond B, we define $\tilde{I}_B = \frac{1}{2} I_B$ if B is of Type II, and $\tilde{I}_B = I_B$ if B is of Type I or Type III; $\tilde{I}_B$ is called the *reduced indicator* of B. For a directed bond (B, ε_B), $[\varepsilon, \varepsilon_B]\tilde{I}_B$ is called the *reduced characteristic vector* of (B, ε_B). The integral span of reduced characteristic vectors of all directed bonds is called the *reduced bond lattice* of (Σ, ε), denoted $\tilde{B}(\Sigma, \varepsilon; \mathbb{Z})$.

Theorem 3.1. (a) *The characteristic vector of any directed cut is a sum of characteristic vectors of some minimal directed cuts.*

(b) *If (U, ε_U) is a minimal directed cut, but is not a directed bond, then there exist two directed bonds (B_1, ε_U) and (B_2, ε_U) such that*

$$[\varepsilon, \varepsilon_U]I_U = \frac{1}{2}[\varepsilon, \varepsilon_U]I_{B_1} + \frac{1}{2}[\varepsilon, \varepsilon_U]I_{B_2}. \tag{3.4}$$

Proof. (a) It follows directly from Theorem 2.5.

(b) Let $U = [X, X^c] \cup E_X$. By Theorem 2.4, let $\Sigma_1, \ldots, \Sigma_k$ be the balanced components of $\Sigma(X^c)(k \geq 1)$, and let $\Sigma'_1, \ldots, \Sigma'_l$ be the unbalanced components of $\Sigma(X^c)$. Since switching does not change characteristic vectors, we may assume that all edges of $\Sigma(X) - E_X$ and Σ_i $(1 \leq i \leq k)$ are positive, the edges of E_X are negative, and ε_U is the positive direction of U (i.e., arrows of edges in U point away from X).

Let E^- and E^+ be the sets of positive and negative edges of $[X, \cup\Sigma_i]$, respectively. Then $B_1 = [X, \cup\Sigma'_i] \cup E_X \cup E^-$ and $B_2 = [X, \cup\Sigma'_i] \cup E_X \cup E^+$ are two directed bonds with the direction ε_U. Thus

$$[\varepsilon, \varepsilon_U]I_U = \frac{1}{2}[\varepsilon, \varepsilon_U]I_{B_1} + \frac{1}{2}[\varepsilon, \varepsilon_U]I_{B_2}. \quad \square$$

Let (W, ε_W) be a directed walk with the sequence $u_0x_0u_1x_1 \ldots x_nu_{n+1}$, where $x_i = u_iu_{i+1}$. We may think of the walk W as a multiset $\{x_0, x_1, \ldots, x_n\}$ of $n+1$ elements. The *characteristic vector* of (W, ε_W) is a function $f_W : E \to \mathbb{Z}$, defined for $x \in E$ by

$$f_W(x) := \sum_{y \in W,\, y = x} [\varepsilon, \varepsilon_W](y). \tag{3.5}$$

For any Abelian group A and function $g : E \to A$, we have

$$\langle f_W, g\rangle := \sum_{x \in E} f_W(x)g(x) = \sum_{x \in W} [\varepsilon, \varepsilon_W](x)g(x). \tag{3.6}$$

Lemma 3.2. *Let $U = [X, X^c] \cup E_X$ be a directed bond of (Σ, ε), i.e., (U, ε) is a directed cut. Let W be a directed walk with direction ε_W. If $x_0, x_k \in U$ and $W = u_0x_0Qu_kx_k$, where Q is a sub-walk (may be empty) inside $\Sigma(X) - E_X$, then $[\varepsilon, \varepsilon_W](x_0)$ and $[\varepsilon, \varepsilon_W](x_k)$ have opposite signs.*

Proof. The sub-walk $Q = u_1x_1u_2x_2 \cdots x_{k-1}u_k$ with the orientation ε_W is a directed walk whose edges have positive sign. Then $\varepsilon_W(u_1, x_1) \neq \varepsilon_W(u_k, x_{k-1})$. Consequently, $\varepsilon_W(u_1, x_0) \neq \varepsilon_W(u_k, x_k)$. Note that $\varepsilon(u_1, x_0) = \varepsilon(u_k, x_k) = 1$ for (U, ε) is a directed cut. By the definition of coupling, we have $[\varepsilon, \varepsilon_W](x_0) \neq [\varepsilon, \varepsilon_W](x_k)$. $\square$

Lemma 3.3. *Let (W, ε_W) be a directed closed positive walk, and let (U, ε_U) be a directed cut. Then f_W is orthogonal to the cut characteristic vector $[\varepsilon, \varepsilon_U]I_U$, i.e.,*

$$\langle f_W, [\varepsilon, \varepsilon_U]I_U\rangle = \sum_{x \in W} [\varepsilon_U, \varepsilon_W](x)I_U(x) = 0. \tag{3.7}$$

Proof. Let $U = [X, X^c] \cup E_X$. Since switching does not change any circuit, bond, and coupling, we may assume positive orientation ε_U and $\varepsilon_U = \varepsilon$ on $E(X) \cup U$. Then $\sigma|_{E(X)-E_X} = 1$,

$\sigma|_{E_X} = -1$, and $\varepsilon(v, e) = 1$ for all incident pairs $(v, e) \in X \times U$. For any sub-walk W' of W, let

$$I(W') := \sum_{x \in W'} [\varepsilon, \varepsilon_W](x) I_U(x). \tag{3.8}$$

Then $I(W) = \langle f_W, [\varepsilon, \varepsilon_U] I_U \rangle$. It suffices to show that $I(W) = 0$.

Case 1: $W \subseteq \Sigma(X^c)$. Since U and $\Sigma(X^c)$ are disjoint, then $W \cap U = \emptyset$. We have $I(W) = 0$ trivially.

Case 2: $W \subseteq \Sigma(X)$. Write $W = P_1 Q_1 P_2 Q_2 \cdots P_k Q_k$, where P_i are sub-walks inside E_X and Q_i are sub-walks inside $E(X) - E_X$. If $W \subseteq E_X$, then $k = 1$, $W = P_1$, and $Q_1 = \emptyset$. If $W \subseteq E(X) - E_X$, then $k = 1$, $W = Q_1$, and $P_1 = \emptyset$. Since all edges of E_X are negative, the orientation ε_W is alternating on P_i; consequently, $[\varepsilon, \varepsilon_W]$ is alternating on P_i. Note that by Lemma 3.2, $[\varepsilon, \varepsilon_W]$ has opposite signs at the end edge of P_i and the initial edge of P_{i+1}, where $P_{k+1} = P_1$. Thus, contracting the edges of all Q_i, $[\varepsilon, \varepsilon_W]$ is alternating on the closed sub-walk $W_U = P_1 P_2 \cdots P_k$. It is clear that $I(W) = I(W_U) = 0$.

Case 3: $W \cap [X, X^c] \neq \emptyset$. Start with a vertex $u_0 \in V(X^c)$ and an edge $x_0 \in [X, X^c] \cap C$ incident with u_0; and travel along the walk W. We break W into some sub-walks W_i inside $E(X) \cup [X, X^c]$ and sub-walks W_i' inside $E(X^c)$, $W = W_1 W_1' W_2 W_2' \cdots W_k W_k'$. It is enough to show that $I(W_i) = 0$. Let each W_i be written as

$$W_i = u_i x_i Q_{0,i} P_{1,i} Q_{1,i} P_{2,i} Q_{2,i} \cdots P_{n_i,i} Q_{n_i,i} v_i y_i,$$

where $x_i, y_i \in [X, X^c]$, $P_{j,i}$ are sub-walks inside E_X, and $Q_{j,i}$ are sub-walks inside $E(X) - E_X$. Similarly, $[\varepsilon, \varepsilon_W]$ is alternating on $P_{j,i}$, for the edges of E_X are negative. By Lemma 3.2, $[\varepsilon, \varepsilon_W]$ has opposite signs at the end edge of $P_{j,i}$ and the initial edge of $P_{j+1,i}$, where $P_{0,i} = x_0$, $P_{n_i+1,i} = x_k$, $0 \le j \le n_i$. Contracting the edges of all $Q_{j,i}$, it follows that $[\varepsilon, \varepsilon_W]$ is alternating along the closed walk $W_{U,i} = u_i x_i W_{U,i}' v_i y_i$, where

$$W_{U,i}' = P_{1,i} P_{2,i} \cdots P_{n_i,i} = u_{1,i} x_{1,i} \cdots u_{k,i} x_{k,i}.$$

If $|W_{U,i}'|$ is even, then $[\varepsilon, \varepsilon_W](x_i) = -[\varepsilon, \varepsilon_W](y_i)$. Thus $I(W_i)$ equals

$$I(W_{U,i}) = [\varepsilon, \varepsilon_U](x_i) + [\varepsilon_C, \varepsilon](y_i) + 2 \sum_{x \in W_{U,i}'} [\varepsilon, \varepsilon_W](x) = 0.$$

If $|W_{U,i}'|$ is odd, we must have $[\varepsilon, \varepsilon_W](x_i) = [\varepsilon, \varepsilon_W](y_i) = -[\varepsilon, \varepsilon](x_{1,i})$. Set

$$W_{U,i}'' = u_{1,i} x_{1,i} u_{2,i} x_{2,i} \cdots u_{k,i} x_{k,i}.$$

We see that $I(W_i)$ equals

$$I(W_{U,i}) = [\varepsilon, \varepsilon_W](x_i) + [\varepsilon, \varepsilon_W](y_i) + 2[\varepsilon, \varepsilon_W](x_{1,i}) + 2 \sum_{x \in W_{U,i}''} [\varepsilon, \varepsilon_W](x) = 0. \quad \square$$

Lemma 3.4. *Each directed circuit (C, ε_C) can be written as a directed closed positive walk (W, ε_W) with $\varepsilon_W = \varepsilon_C$. Moreover, $f_W = [\varepsilon, \varepsilon_C] I_C$.*

Proof. It is easy to write (C, ε_C) as a directed closed positive walk (W, ε_W) with $\varepsilon_W = \varepsilon_C$. To see $f_W = [\varepsilon, \varepsilon_C] I_C$, let $C = C_1 P C_2$ for example. Since each edge $x \in C \cup C_2$ appears once in W, we have $f_W(x) = [\varepsilon, \varepsilon_W](x) = [\varepsilon, \varepsilon_C](x)$; and since each edge $x \in P$ appears twice in W, we have $f_W(x) = 2[\varepsilon, \varepsilon_W](x) = 2[\varepsilon, \varepsilon_C](x)$. Hence $f_W = [\varepsilon, \varepsilon_C] I_C$. $\square$

Notice that the circuits of Σ form a system of circuits of a matroid on the edge set $E(\Sigma)$. Following the standard concept of matroid, we define an *independent set* of Σ to be an edge set whose induced signed graph does not contain any circuit. A *basis* of Σ is a maximal independent set F, i.e., F is not properly contained in any independent set. When Σ is connected and unbalanced, a basis of Σ may *not* be necessarily connected; however each of its connected components contains a unique unbalanced cycle; see [10]. Let F be a basis of Σ. For each edge $e \in F^c (:= E - F)$, there is a unique circuit C_e contained in $F \cup \{e\}$. Similarly, for each edge $e \in F$, there is a unique bond B_e contained in $F^c \cup \{e\}$.

Theorem 3.5. *The circuit space* $Z(\Sigma, \varepsilon, \mathbb{R})$ *and the bond space* $B(\Sigma, \varepsilon; \mathbb{R})$ *are orthogonal complements in the Euclidean space* $\mathbb{R}^E$. *Moreover,*

$$\dim Z(\Sigma, \varepsilon; \mathbb{R}) = |E| - |V| + b(\Sigma), \tag{3.9}$$

$$\dim B(\Sigma, \varepsilon; \mathbb{R}) = |V| - b(\Sigma). \tag{3.10}$$

Proof. The orthogonality of $Z(\Sigma, \varepsilon; \mathbb{R})$ and $B(\Sigma, \varepsilon; \mathbb{R})$ follows from Lemmas 3.3 and 3.4. To see the dimension formulas, it is enough to show that

$$\dim Z(\Sigma, \varepsilon; \mathbb{R}) \geq |E| - |V| + b(\Sigma),$$
$$\dim B(\Sigma, \varepsilon; \mathbb{R}) \geq |V| - b(\Sigma).$$

Let F be a basis of Σ, i.e., F is a maximal subset of $E(\Sigma)$ such that $\Sigma(F)$ does not contain any circuit. It is known that

$$|F| = |V| - b(\Sigma), \qquad |F^c| = |E| - |V| + b(\Sigma);$$

see [10]. For each edge $x \in F^c$, there is a unique circuit $C_x \subseteq F \cup x$ such that $C_x \cap F^c = \{x\}$. For each $x \in F$, there is a unique bond $B_x \subseteq F^c \cup x$ such that $B_x \cap F = \{x\}$. Then the characteristic vectors of the circuits C_x $(x \in F^c)$ are linearly independent. Similarly, the characteristic vectors of the bonds B_x $(x \in F)$ are linearly independent. The dimension inequalities follow immediately. $\square$

4. Flow and tension spaces (lattices)

The *incidence matrix* of (Σ, ε) is a matrix $\boldsymbol{M}(\Sigma, \varepsilon) = [\boldsymbol{m}(u, x)]$ indexed by $(u, x) \in V \times E$, whose (u, x)-entry is defined by

$$\boldsymbol{m}(u, x) = \begin{cases} \varepsilon(u, x) & \text{if } x \text{ is a non-loop at } u, \\ 2\varepsilon(u, x) & \text{if } x \text{ is a negative loop at } u, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We denote by $\operatorname{Row} \boldsymbol{M}(\Sigma, \varepsilon; \mathbb{R})$ and $\operatorname{Row} \boldsymbol{M}(\Sigma, \varepsilon; \mathbb{Z})$ the real and integral spans of the row vectors of $\boldsymbol{M}(\Sigma, \varepsilon)$, respectively. Let $\operatorname{End}(x)$ denote the multi-set of end vertices of an edge x. Then $\boldsymbol{m}(u, x)$ can be written as

$$\boldsymbol{m}(u, x) = \sum_{v \in \operatorname{End}(x),\, v = u} \varepsilon(v, x). \tag{4.2}$$

Lemma 4.1. *Let* ν *be a switching function on the vertex set* V *of* (Σ, ε). *Then*

$$\boldsymbol{M}(\Sigma^\nu, \varepsilon^\nu) = \boldsymbol{D}^\nu \boldsymbol{M}(\Sigma, \varepsilon),$$

where $\boldsymbol{D}^\nu$ is a diagonal matrix indexed by $V \times V$, whose diagonal entries are $\nu(u)$, $u \in V$.

Proof. It follows from $\varepsilon^\nu(u, x) = \nu(u)\varepsilon(u, x)$. □

A function $f : E(\Sigma) \to A$ is said to be *conservative* at a vertex u if

$$\sum_{x\in E_u} \boldsymbol{m}(u, x) f(x) = \sum_{x\in E} \boldsymbol{m}(u, x) f(x) = 0, \tag{4.3}$$

where E_u is the set of all edges incident with u, and A is an Abelian group. If f is conservative at every vertex of Σ, we call f a *flow* of (Σ, ε) with values in A. The zero function $f \equiv 0$ is called the *zero flow* (or *trivial flow*). Flows with values in $\mathbb{Z}$ ($\mathbb{R}$) are called *integral* (*real*) *flows*. We denote by $F(\Sigma, \varepsilon; A)$ the Abelian group of all flows of (Σ, ε) with values in A, i.e., $F(\Sigma, \varepsilon; A) := \operatorname{Ker} \boldsymbol{M}(\Sigma, \varepsilon; A)$. We are interested in the *flow space* $F(\Sigma, \varepsilon; \mathbb{R})$ and the *flow lattice* $F(\Sigma, \varepsilon; \mathbb{Z})$.

Lemma 4.2. *The characteristic vector f_W of any directed closed positive walk (W, ε_W) is an integral flow of (Σ, ε).*

Proof. Write $W = u_0x_0u_1x_1 \dots x_nu_{n+1}$; fix a vertex $u \in V$; and let $u_{n_1}, u_{n_2}, \dots, u_{n_k}$ denote the vertex u appeared in the sequence of W. Since $\varepsilon_W(u_{n_i}, x_{n_i-1}) + \varepsilon_W(u_{n_i}, x_{n_i}) = 0$, we have

$$\begin{aligned}
\sum_{x\in E} \boldsymbol{m}(u, x) f_W(x) &= \sum_{x\in W} \boldsymbol{m}(u, x)[\varepsilon, \varepsilon_W](x) \quad \text{[By (4.2)]}\\
&= \sum_{x\in W} \sum_{v\in \mathrm{End}(x), v=u} \varepsilon(v, x)[\varepsilon, \varepsilon_W](x)\\
&= \sum_{x\in W} \sum_{v\in \mathrm{End}(x), v=u} \varepsilon_W(v, x)\\
&= \sum_{i=1}^{k} \left[\varepsilon_W(u_{n_i}, x_{n_i-1}) + \varepsilon_W(u_{n_i}, x_{n_i})\right] = 0.
\end{aligned}$$

Hence f_W is an integral flow of (Σ, ε). □

Lemma 4.3. $Z(\Sigma, \varepsilon, \mathbb{Z}) \subseteq F(\Sigma, \varepsilon; \mathbb{Z})$

Proof. It follows from Lemmas 3.4 and 4.2. □

A *tension* of (Σ, ε) with values in an Abelian group A is a function $g : E \to A$ such that for any directed circuit (C, ε_C),

$$\langle[\varepsilon, \varepsilon_C]I_C, g\rangle = \sum_{x\in C} [\varepsilon, \varepsilon_C]I_C(x)g(x) = 0. \tag{4.4}$$

We denote by $T(\Sigma, \varepsilon; A)$ the Abelian group of all tensions of (Σ, ε) with values in A. We are interested in the *tension space* $T(\Sigma, \varepsilon; \mathbb{R})$ and the *tension lattice* $T(\Sigma, \varepsilon; \mathbb{Z})$.

Lemma 4.4. $B(\Sigma, \varepsilon; \mathbb{R}) = T(\Sigma, \varepsilon; \mathbb{R})$

Proof. The tension space $T(\Sigma, \varepsilon; \mathbb{R})$ and the bond space $B(\Sigma, \varepsilon; \mathbb{R})$ are orthogonal complements of $Z(\Sigma, \varepsilon; \mathbb{R})$ in $\mathbb{R}^{E(\Sigma)}$ by definition and by Theorem 3.5. So they are the same. □

Lemma 4.5. (a) *Let (B, ε_B) be a directed bond with $B = [X, X^c] \cup E_X$, and let ν_X be the bond switching function. Then*

$$[\varepsilon, \varepsilon_B] I_B = \sum_{u \in X} \boldsymbol{m}_u^{\nu_X},$$

where $\boldsymbol{m}_u^{\nu_X}$ is the row vector of $\boldsymbol{M}^{\nu_X}(\Sigma, \varepsilon)$ indexed by a vertex u.

(b) *The row vector $\boldsymbol{m}_u$ of $\boldsymbol{M}(\Sigma, \varepsilon)$ indexed by a vertex u is a linear combination of characteristic vectors of some directed bonds with coefficients $\pm 1, \pm 2$.*

Proof. (a) Notice that $\varepsilon_B^{\nu_X}(u, e) = 1$ for all incident pairs (u, e) with $u \in X$ and $e \in B$.

Case 1: The bond B is of Type II, i.e., $B = [X, X^c] \cup E_X$ with $X = V$. For each edge $e = uv \in E_X$, its two vertices u, v must belong to X. Then

$$\begin{aligned} \text{LHS} &= 2\,[\varepsilon, \varepsilon_B](e) = 2\,[\varepsilon^{\nu_X}, \varepsilon_B^{\nu_X}] \\ &= 2\,\varepsilon^{\nu_X}(u, e) = \sum_{w \in X} \boldsymbol{m}_{wx}^{\nu_X} = \text{RHS}. \end{aligned}$$

For any edge $x = uv \notin E_X$, the LHS is obviously zero, while the RHS is also zero.

Case 2: The bond B is of Type I or Type III. For any edge $e = uv \in [X, X^c]$, one of the vertices u, v belongs to X, say $u \in X$. Then

$$\begin{aligned} \text{LHS} &= [\varepsilon, \varepsilon_B](e) = [\varepsilon^{\nu_X}, \varepsilon_B^{\nu_X}] \\ &= \varepsilon^{\nu_X}(u, e) = \sum_{w \in X} \boldsymbol{m}_{we}^{\nu_X} = \text{RHS}. \end{aligned}$$

For an edge $e = uv \in E_X$, if e is a loop, then $u = v$, and

$$\begin{aligned} \text{LHS} &= 2[\varepsilon, \varepsilon_B](e) = 2[\varepsilon^{\nu_X}, \varepsilon_B^{\nu_X}](e) \\ &= 2\varepsilon^{\nu_X}(u, e) = \sum_{w \in X} \boldsymbol{m}_{we}^{\nu_X} = \text{RHS}; \end{aligned}$$

If $e = uv$ is not a loop, then $u \neq v$, and

$$\begin{aligned} \text{LHS} &= 2[\varepsilon, \varepsilon_B](e) = 2[\varepsilon^{\nu_X}, \varepsilon_B^{\nu_X}](e) \\ &= \varepsilon^{\nu_X}(u, e) + \varepsilon^{\nu_X}(v, e) = \sum_{w \in X} \boldsymbol{m}_{we}^{\nu_X} = \text{RHS}. \end{aligned}$$

For edges $e \notin B$, the LHS is zero. If $e \in E(X^c)$, the right-hand side is also obviously zero as the end vertices of e cannot be in X. If $e \in E(X) - E_X$, then $e = uv$ is a positive edge and $u, v \in X$; clearly, $\varepsilon^{\nu_X}(u, e) + \varepsilon^{\nu_X}(v, e) = 0$; so the RHS is zero.

(b) Let the components of $\Sigma(V - u)$ be partitioned into balanced components Σ_i, Σ'_j, and unbalanced components Σ''_k, such that $\Sigma_i \cup [\{u\}, V(\Sigma_i)]$ are balanced, and $\Sigma'_j \cup [\{u\}, V(\Sigma'_j)]$ are unbalanced. Let ν be a switching function such that the edges in $\Sigma_i \cup [\{u\}, V(\Sigma_i)]$ and Σ'_j are all positive; the switching function ν can be made to only switch vertices inside Σ_i and Σ'_j. Then the edge sets

$$B_i = [V(\Sigma_i), \{u\}], \qquad B'_j = \Big[V(\Sigma'_j), \{u\}\Big], \qquad B_0 = \left[\{u\}, \bigcup_k V(\Sigma''_k)\right] \cup E_u^-$$

are bonds of Σ^ν, where E_u^- is the set of negative edges in $\Sigma^\nu \left(\{u\} \cup \bigcup_j V(\Sigma'_j)\right)$.

Let $\varepsilon_i, \varepsilon'_j$, and ε_0 be positive directions on the bonds B_i, B'_j, and B_0, respectively, i.e., $\varepsilon_i(u_i, e_i) = 1$ for $u_i \in V(\Sigma_i)$ and $e_i \in B_i$, $\varepsilon'_j(u'_j, e'_j) = 1$ for $u'_j \in V(\Sigma'_j)$ and $e'_j \in B'_j$, and $\varepsilon_0(u, e_0) = 1$ for $e_0 \in B_0$. Let $0 \le i \le l$, $0 \le j \le m$, and $0 \le k \le n$. Notice that the row vector $\boldsymbol{m}_u$ of $\boldsymbol{M}$ equals the row vector $\boldsymbol{m}_u^\nu$ of $\boldsymbol{M}^\nu$ as the vertex u is not switched by ν.

Case 1: $n \neq 0$. We then have

$$\boldsymbol{m}_u = [\varepsilon^\nu, \varepsilon_0] I_{B_0} - \sum_i [\varepsilon^\nu, \varepsilon_i] I_{B_i} - \sum_j [\varepsilon^\nu, \varepsilon'_j] I_{B'_j}.$$

Case 2: $n = 0$ *and* $m \neq 1$, *or at least one negative loop exists at* u. Then

$$\boldsymbol{m}_u = 2[\varepsilon^\nu, \varepsilon_0] I_{B_0} - \sum_i [\varepsilon^\nu, \varepsilon_i] I_{B_i} - \sum_j [\varepsilon^\nu, \varepsilon'_j] I_{B'_j}.$$

Case 3: $n = 0, m = 1$, *and there is no negative loop at* u. Then

$$\boldsymbol{m}_u = [\varepsilon^\nu, \varepsilon_{B^+}] I_{B^+} + [\varepsilon^\nu, \varepsilon_{B^-}] I_{B^-} - \sum_i [\varepsilon^\nu, \varepsilon_i] I_{B_i},$$

where B^+ is set of positive edges in Σ'_1 and ε_{B^+} is the positive direction on B^+, and B^- is the set negative edges in Σ'_1 and ε_{B^-} is the positive direction on B^-. $\square$

Theorem 4.6. $B(\Sigma, \varepsilon; \mathbb{Z}) = \mathrm{Row}\, \boldsymbol{M}(\Sigma, \varepsilon; \mathbb{Z})$.

Proof. It follows from Lemma 4.5. $\square$

Theorem 4.7. *The vector spaces* $F(\Sigma, \varepsilon; \mathbb{R})$ *and* $T(\Sigma, \varepsilon; \mathbb{R})$ *are orthogonal complements in* $\mathbb{R}^E$*; and* $F(\Sigma, \varepsilon; \mathbb{R}) = Z(\Sigma, \varepsilon; \mathbb{R})$.

Proof. A flow f is equivalent to $\boldsymbol{M}(\Sigma, \varepsilon) f = 0$. So $F(\Sigma, \varepsilon; \mathbb{R}) = \mathrm{Ker}\, \boldsymbol{M}(\Sigma, \varepsilon; \mathbb{R})$. Since $\mathrm{Ker}\, \boldsymbol{M}(\Sigma, \varepsilon; \mathbb{R})$ is the orthogonal complement of $\mathrm{Row}\, \boldsymbol{M}(\Sigma, \varepsilon; \mathbb{R})$ in $\mathbb{R}^E$, it follows that $\mathrm{Ker}\, \boldsymbol{M}(\Sigma, \varepsilon; \mathbb{R}) = Z(\Sigma, \varepsilon; \mathbb{R})$. $\square$

Recall that when Σ is connected and unbalanced, a basis of Σ is not necessarily connected; however, each of its connected components contains a unique unbalanced cycle; see [10]. The following lemma shows when a connected basis exists and what it looks like.

Lemma 4.8. *Let* Σ *be connected and unbalanced. Then there exists a connected basis for* Σ. *Furthermore, let* F *be a connected basis of* Σ.

(a) *If* $e \in F^c$ *and the unique circuit* C_e *of* $F \cup \{e\}$ *is of Type III, then the edge* e *is not on the circuit path of* C_e.
(b) *If* $e \in F$ *and the unique bond* B_e *of* $F^c \cup \{e\}$ *is of Type III, then the edge* e *is not in the bond core of* B_e.

Proof. The existence of bases for Σ is obvious. Let F be a basis of Σ. If F is disconnected, there is an edge e between two components of F. Then the unique circuit C_e in $F \cup \{e\}$ must be of Type III. Remove an edge from C_e that is not on the circuit path; a basis of Σ is obtained with a smaller number of components. Continue this procedure; a connected basis is finally constructed.

(a) Suppose there is an edge $e \in F^c$ such that the unique circuit C_e of $F \cup \{e\}$ is of Type III, and e is on the circuit path. Then F contains two vertex disjoint unbalanced cycles. Since F is connected, there is a path between the two unbalanced cycles to form a circuit of Type III in F. This is a contradiction.

(b) Similarly, suppose there is an edge $e \in F$ such that the unique bond B_e of $F^c \cup e$ is of Type III, and e is in the bond core E_X. Let $B_e = [X, X^c] \cup E_X$. Then the component of F that contains the edge e is unbalanced, and must be contained in $\Sigma(X)$. Since F is connected, it follows that F is contained in $\Sigma(X)$; so $X = V$. This is contrary to the assumption that B_e is a bond of Type III. $\square$

Let F be a basis of Σ. If a flow is zero on F^c, then the flow is the zero flow. Analogously, if a tension is zero on F, then the tension is the zero tension. Whenever Σ is given a basis, the explicit descriptions for flows and tensions are given by the following theorem, which is well known for graphs; see [3].

Theorem 4.9. *Let F be a basis of (Σ, ε). For each edge $e \in E(\Sigma)$, let ε_e denote a direction of the unique circuit C_e if $e \in F^c$, and denote a direction of the unique bond B_e if $e \in F$. Then for any flow f and tension g of (Σ, ε),*

$$f = \sum_{e \in F^c} \frac{[\varepsilon, \varepsilon_e](e) f(e)}{I_{C_e}(e)} [\varepsilon, \varepsilon_e] I_{C_e}, \tag{4.5}$$

$$g = \sum_{e \in F} \frac{[\varepsilon, \varepsilon_e](e) g(e)}{I_{B_e}(e)} [\varepsilon, \varepsilon_e] I_{B_e}. \tag{4.6}$$

In particular, if each component of F is a basis of a component of Σ, then

$$f = \sum_{e \in F^c} [\varepsilon, \varepsilon_e](e) f(e) [\varepsilon, \varepsilon_e] I_{C_e}, \tag{4.7}$$

$$g = \sum_{e \in F} [\varepsilon, \varepsilon_e](e) g(e) [\varepsilon, \varepsilon_e] \tilde{I}_{B_e}, \tag{4.8}$$

where $\tilde{I}_{B_e}$ is the reduced indicator of B_e.

Proof. Both sides of (4.5) are flows, and they agree on the set F^c. So they agree on the whole set $E(\Sigma)$. Similarly, both sides of (4.6) are tensions, and they agree on the set F. So they agree on the whole set $E(\Sigma)$.

By Parts (b) and (c) of Lemma 4.8, we have $I_{C_e}(e) = 1$ and $I_{B_e}(e) = 1$ in (4.5) and (4.6), respectively. So (4.7) and (4.8) follow immediately. $\square$

Corollary 4.10. $Z(\Sigma, \varepsilon; \mathbb{Z}) = F(\Sigma, \varepsilon; \mathbb{Z})$ *and* $\tilde{B}(\Sigma, \varepsilon; \mathbb{Z}) = T(\Sigma, \varepsilon; \mathbb{Z})$.

Proof. By Lemma 4.3, we only need to show that $F(\Sigma, \varepsilon; \mathbb{Z}) \subseteq Z(\Sigma, \varepsilon; \mathbb{Z})$. This is readily shown, for every integral flow is an integral linear combination of circuit characteristic vectors, by (4.7).

Similarly, (4.8) shows that $T(\Sigma, \varepsilon; \mathbb{Z}) \subseteq \tilde{B}(\Sigma, \varepsilon; \mathbb{Z})$. Since $B(\Sigma, \varepsilon; \mathbb{R}) = T(\Sigma, \varepsilon; \mathbb{R})$ by Lemma 4.4 and $T(\Sigma, \varepsilon; \mathbb{Z}) = T(\Sigma, \varepsilon; \mathbb{R}) \cap \mathbb{Z}^E$, we see that $\tilde{B}(\Sigma, \varepsilon; \mathbb{Z}) \subseteq T(\Sigma, \varepsilon; \mathbb{Z})$. $\square$

5. Relation between colorings and tensions

Let A be an Abelian group. A *coloring* (or *potential*) of the signed graph Σ with a color set A is a function $f : V \to A$. A coloring f is said to be *proper* if

$$f(v) \neq \sigma(e) f(u) \tag{5.1}$$

for any edge $x = uv$ with end vertices u and v. We denote by $K(\Sigma, A)$ the set of all colorings of Σ with the color set A, and by $K_{\mathrm{nz}}(\Sigma, A)$ the set of all proper colorings. There is a *difference operator* $\delta : A^V \to A^E$, defined for functions $f \in A^V$ by

$$(\delta f)(x) = \varepsilon(u, x) f(u) + \varepsilon(v, x) f(v), \quad x = uv. \tag{5.2}$$

The following Theorem 5.1 states the relation between colorings and tensions of signed graphs, similar to that of ordinary graphs [3].

Theorem 5.1. *Let R be a commutative ring with the unity 1 and the element 2 invertible. Then $\delta : R^V \to T(\Sigma, \varepsilon; R)$ is an R-module epimorphism with* $\operatorname{Ker} \delta \simeq R^{b(\Sigma)}$.

Proof. Let $f \in R^V$. We first show that δf is a tension. It is enough to show that $\langle f_W, \delta f\rangle = 0$ for any directed closed positive walk (W, ε_W). Let $W = u_0 x_0 u_1 x_1 \dots u_n x_n u_{n+1}$, where $u_{n+1} = u_0$, $x_{n+1} = x_0$, and $\varepsilon_W(u_i, x_{i-1}) + \varepsilon_W(u_i, x_i) = 0$. Then

$$\begin{aligned}
\langle f_W, \delta f\rangle &= \sum_{x\in W} [\varepsilon, \varepsilon_W](x)(\delta f)(x) = \sum_{i=0}^{n} [\varepsilon, \varepsilon_W](x_i)(\delta f)(x_i) \\
&= \sum_{i=0}^{n} [\varepsilon, \varepsilon_W](x_i) \left[\varepsilon(u_i, x_i) f(u_i) + \varepsilon(u_{i+1}, x_i) f(u_{i+1})\right] \\
&= \sum_{i=0}^{n} \left[\varepsilon_W(u_i, x_i) f(u_i) + \varepsilon_W(u_{i+1}, x_i) f(u_{i+1})\right] \\
&= \sum_{i=0}^{n} \varepsilon_W(u_i, x_i) f(u_i) + \sum_{i=1}^{n+1} \varepsilon_W(u_i, x_{i-1}) f(u_i) = 0.
\end{aligned}$$

Next we show that δ is surjective. Let g be a tension of (Σ, ε). We construct a potential f such that $\delta f = g$. Fix a vertex u_0 and assume that $f(u_0)$ is given. We define f at an arbitrary vertex u as follows: Take a walk W from u_0 to u; write $W = u_0 x_0 u_1 x_1 \dots u_m x_m u_{m+1}$, where $u_{m+1} = u$, $x_i = u_i u_{i+1}$; and choose a direction ε_W, i.e., $\varepsilon_W(u_i, x_{i-1}) + \varepsilon_W(u_i, x_i) = 0$. To have $\delta f = g$, whenever $f(u_i)$ is given, the value $f(u_{i+1})$ must be given by

$$(\delta f)(x_i) = \varepsilon(u_i, x_i) f(u_i) + \varepsilon(u_{i+1}, x_i) f(u_{i+1}) = g(x_i).$$

Since $\varepsilon(u_i, x_i)\varepsilon(u_{i+1}, x_i) = -\sigma(x_i)$, we obtain the recurrence relation

$$f(u_{i+1}) = \sigma(x_i) f(u_i) + \varepsilon(u_{i+1}, x_i) g(x_i), \quad 0 \le i \le m. \tag{5.3}$$

Since (W, ε_W) is directed, we have

$$\prod_{i=0}^{k} \sigma(x_i) = -\varepsilon_W(u_{k+1}, x_k)\varepsilon_W(u_0, x_0), \quad 0 \le k \le m.$$

The recurrence relation (5.3) implies that

$$\begin{aligned}
f(u_{m+1}) &= f(u_0) \prod_{i=0}^{m} \sigma(x_i) + \sum_{i=0}^{m} \varepsilon(u_{i+1}, x_i) g(x_i) \prod_{j=i+1}^{m} \sigma(x_j) \\
&= \sigma(W) \left[f(u_0) + \sum_{i=0}^{m} \varepsilon(u_{i+1}, x_i) g(x_i) \prod_{j=0}^{i} \sigma(x_j) \right]
\end{aligned}$$

$$= \sigma(W)\left[f(u_0) - \varepsilon_W(u_0, x_0) \sum_{i=0}^{m} \varepsilon(u_{i+1}, x_i)\varepsilon_W(u_{i+1}, x_i) g(x_i) \right].$$

Thus, whenever $f(u_0)$ is given, the value $f(u) = f(u_{m+1})$ must be defined by

$$f(u_{m+1}) := \sigma(W)\left[f(u_0) - \varepsilon_W(u_0, x_0) \sum_{i=0}^{m} [\varepsilon, \varepsilon_W](x_i) g(x_i) \right]. \tag{5.4}$$

We are left to show that f is well defined. It suffices to show that, when W is a closed walk, we should have

$$f(u_0) = \sigma(W)\left[f(u_0) - \varepsilon_W(u_0, x_0) \sum_{i=0}^{m} [\varepsilon, \varepsilon_W](x_i) g(x_i) \right]. \tag{5.5}$$

If the closed walk W is positive, i.e, $\sigma(W) = 1$, then $\sum_{i=0}^{m} [\varepsilon, \varepsilon_W](x_i) g(x_i) = 0$, for g is a tension. The identity (5.5) holds automatically.

Case 1: *The component Σ_0 that contains u_0 is balanced.* Since every closed walk of Σ_0 has positive sign, the identity (5.5) holds for every closed walk with direction. Hence the value of f at a chosen base vertex u_0 can be arbitrarily assigned, and the values at other vertices are uniquely determined.

Case 2: *The component Σ_0 that contains u_0 is unbalanced.* Let the closed walk W have negative sign, i.e., $\sigma(W) = \prod_{i=1}^{m} \sigma(x_i) = -1$, $u_{m+1} = u_0$. Then (5.4) implies

$$f(u_0) = f(u_{m+1}) = \frac{\varepsilon_W(u_0, x_0)}{2} \sum_{i=0}^{m} [\varepsilon, \varepsilon_W](x_i) g(x_i). \tag{5.6}$$

For each vertex $u_0 \in V(\Sigma_0)$, we define $f(u_0)$ by (5.6) with any directed closed walk W initiating at u_0 with negative sign. We need to show that f is well defined at u_0.

Let W' be another closed walk with negative sign, having a direction $\varepsilon_{W'}$ initiating at u_0 and the vertex-edge sequence $v_0 y_0 v_1 y_1 \ldots v_n y_n v_{n+1}$, where $v_{n+1} = v_0 = u_0$. Let f_W be defined by (5.4) with the directed closed walks W and W', respectively. It suffices to show that $f_W(u_0) = f_{W'}(v_0)$, where

$$f_{W'}(v_0) = f_{W'}(v_{n+1}) = \frac{\varepsilon_{W'}(v_0, y_0)}{2} \sum_{i=0}^{n} [\varepsilon, \varepsilon_{W'}](y_i) g(y_i).$$

Let us write $\varepsilon_{W'}(v_0, y_0) = \theta \varepsilon_W(u_{m+1}, x_m)$, where $\theta = \pm 1$. Then

$$\varepsilon_W(u_{m+1}, x_m) = \theta \varepsilon_{W'}(v_0, y_0) = -(-\theta \varepsilon_{W'})(v_0, y_0).$$

Since both W and W' are negative closed walks, we have

$$\varepsilon_W(u_0, x_0) = \varepsilon_W(u_{m+1}, x_m), \qquad \varepsilon_{W'}(v_0, y_0) = \varepsilon_{W'}(v_{n+1}, y_n).$$

It follows that $\varepsilon_{W'}(v_{n+1}, y_n) = \theta \varepsilon_W(u_0, x_0)$, i.e., $(-\theta \varepsilon_{W'})(v_{n+1}, y_n) = -\varepsilon_W(u_0, x_0)$. This means that the concatenation

$$(WW', \varepsilon_{W\,W'}) := (W, \varepsilon_W)(W', -\theta \varepsilon_{W'})$$

is a directed closed positive walk, whose vertex-edge sequence is

$$u_0 x_0 u_1 x_1 \cdots u_m x_m u_{m+1}(v_0) y_0 v_1 y_1 \cdots v_n y_n v_{n+1},$$

and whose direction $\varepsilon_{W\,W'}$ is defined by

$$\varepsilon_{W\,W'}(u,x)=\begin{cases}\varepsilon_W(u,x) & \text{if } x\in W,\\ -\theta\varepsilon_{W'}(u,x) & \text{if } x\in W',\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\varepsilon_{W'}(v_0,y_0)=\theta\varepsilon_W(u_0,x_0)$. We thus have

$$\begin{aligned} f_W(u_0)-f_{W'}(v_0) &= \frac{\varepsilon_W(u_0,x_0)}{2}\left(\sum_{i=0}^{m}[\varepsilon,\varepsilon_W](x_i)g(x_i)-\theta\sum_{j=0}^{n}[\varepsilon,\varepsilon_{W'}](y_j)g(y_j)\right)\\ &= \frac{\varepsilon_W(u_0,x_0)}{2}\sum_{x\in WW'}[\varepsilon,\varepsilon_{W\,W'}](x)g(x)\\ &= \frac{\varepsilon_W(u_0,x_0)}{2}\langle f_{W\,W'},g\rangle=0. \quad \text{(By Lemma 4.2)}\end{aligned}$$

Finally, for $f\in R^V$, if $\delta f=0$, then $f(u)=f(v)$ for positive edges $x=uv$, and $f(u)=-f(v)$ for negative edges $x=uv$. We see that f must be constant on each balanced component of Σ, and must be zero on each unbalanced component. Hence $\operatorname{Ker}\delta\simeq R^{b(\Sigma)}$. □

Corollary 5.2. *Let R be a commutative ring with the unity 1 and the element 2 invertible. Then*

$$R^{V(\Sigma)}\simeq R^{b(\Sigma)}\oplus T(\Sigma,\varepsilon;R).$$

In particular, if q is an odd integer, then

$$(\mathbb{Z}/q\mathbb{Z})^{V(\Sigma)}\simeq(\mathbb{Z}/q\mathbb{Z})^{b(\Sigma)}\oplus T(\Sigma,\varepsilon;\mathbb{Z}/q\mathbb{Z}).$$

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